Continued Radicals and Cantor Sets

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Western Kentucky University
KYMAA Annual Meeting

March 31, 2012
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- Continued Fractions
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- Infinite Products
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- Continued Fractions
- Series
- Infinite Products
- Continued Radicals
"Arithmetica infinitorum", Wallis (1653)
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\[ x = a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \ldots}}} \]
\[ \sqrt{x_0} + \sqrt{x_1} + \sqrt{x_2} + \sqrt{\ldots} + \sqrt{x_n} \]
Continued Radicals and Cantor Sets

\[ \lim_{n \to \infty} \sqrt{x_0 + \sqrt{x_1 + \sqrt{x_2 \ldots + \sqrt{x_n}}} \]
Introduction
Continued Radical Examples
Convergence of Continued Radicals
Computing $\phi_1$
Continued Radicals Forming Cantor Set
Measure
Conclusion

Iterated Function System
Continued Fractions
Nested Radicals
Continued Radicals
Cantor Ternary Set

0  $\frac{1}{3}$  $\frac{2}{3}$  1
Continued Radicals Forming Cantor Set

Measure

Conclusion

Cantor Ternary Set
Bridges and Gaps
Continued Radical Examples
Convergence of Continued Radicals
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Golden Ratio
3, Ramanujan

$$\phi_1 = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}$$
\[
\phi_1 = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}}
\]

\[
= \frac{1 + \sqrt{5}}{2}
\]
3 = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \ldots}}}}
Difference Equations

\[ 3 = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \ldots}}}} \]
Proposition

If the sequence \((a_i)_{i=1}^{\infty}\) of nonnegative numbers is bounded above, then \(\sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + \ldots}}}\) converges. (Johnson & Richmond, 2008)
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Convergence of $\phi_1$
Solving for $\phi_1$

$\sqrt{1} = 1$
$\sqrt{1 + \sqrt{1}} = \sqrt{1 + 1} = \sqrt{2}$
$\sqrt{1 + \sqrt{1 + \sqrt{1}}} = \sqrt{1 + \sqrt{2}}$
...

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$\sqrt{1} = 1$
\[ \sqrt{1} = 1 \]

\[ \sqrt{1} + \sqrt{1} = \sqrt{1} + 1 = \sqrt{2} \]
\[ \sqrt{1} = 1 \]
\[ \sqrt{1 + \sqrt{1}} = \sqrt{1 + 1} = \sqrt{2} \]
\[ \sqrt{1 + \sqrt{1} + \sqrt{1}} = \sqrt{1 + \sqrt{1 + 1}} = \sqrt{1 + \sqrt{2}} \]
\[
\begin{align*}
\sqrt{1} &= 1 \\
\sqrt{1} + \sqrt{1} &= \sqrt{1 + 1} = \sqrt{2} \\
\sqrt{1 + \sqrt{1} + \sqrt{1}} &= \sqrt{1 + \sqrt{1} + 1} = \sqrt{1 + \sqrt{2}} \\
\ldots
\end{align*}
\]
The sequence $\phi_1 = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}$ converges.
The sequence $\phi_1 = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}$ converges. How do we compute it?
Let $\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}} = x}.$
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Now square both sides to get,
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$1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}} = x^2}$. 

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Now square both sides to get,

\[
1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}} = x^2.
\]

That is,

\[
1 + x = x^2.
\]
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We can rewrite this as,

$x^2 - x - 1 = 0$. 
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This is just solving a quadratic with \( a = 1 \), \( b = -1 \), and \( c = -1 \). Using the quadratic equation, we get,

\[ x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}. \]
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Using the quadratic equation, we get,

\[
x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}.
\]

Since \( \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}} > 0, \) we have \( x = \frac{1+\sqrt{5}}{2}. \)
Theorem

If \( m_1 \) and \( m_2 \) are natural numbers with \( m_1 < m_2 \), then the set
\[
D = \{ \sqrt{a_1} + \sqrt{a_2} + \sqrt{a_3} + \ldots \mid a_i \in \{m_1, m_2\} \forall i \in \mathbb{N} \}
\]
is homeomorphic to the Cantor ternary set \( \mathcal{C} \). (Johnson & Richmond, 2008)
Theorem

Suppose $M = \{m_1, m_2, \ldots, m_p\} \subseteq \mathbb{N}$ where
$0 < m_1 < m_2 < \ldots < m_p$ and

$$\sqrt{m_i} + \sqrt{m_p} + \sqrt{m_p} + \ldots \geq \sqrt{m_{i+1}} + \sqrt{m_1} + \sqrt{m_1} + \ldots$$

$\forall i \in \{1, \ldots, p - 1\}$.

Then the set of numbers representable as a continued radical
$\sqrt{a_1, a_2, a_3, \ldots}$ with terms $a_i \in M$ is the interval
$\left[\frac{1 + \sqrt{4(m_1) + 1}}{2}, \frac{1 + \sqrt{4(m_p) + 1}}{2}\right]$. 
Example

Continued Radical Using \{1, 2\}.

\[
\frac{\sqrt{5}}{2} + \sqrt{3} + \sqrt{2 + \frac{1 + \sqrt{5}}{2}}
\]
## Example

Continued Radical Using \{1, 2\}.

\[
\frac{1+\sqrt{5}}{2} \quad \sqrt{3} \quad \sqrt{2 + \frac{1+\sqrt{5}}{2}} \quad 2
\]
Example

Continued Radical Using \( \{1, 2\} \).

\[
\begin{array}{c c c c c c c}
\frac{1+\sqrt{5}}{2} & \sqrt{3} & \sqrt{2 + \frac{1+\sqrt{5}}{2}} & 2 \\
\hline
& & & & & & \\
& & & & & & \\
\end{array}
\]
Example

Continued Radical Using \{1, 2\}.

\[
\frac{1+\sqrt{5}}{2} \quad \sqrt{3} \quad \sqrt{2 + \frac{1+\sqrt{5}}{2}} \quad 2
\]
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$\mu(C_3) = 0$
$\mu(C_\epsilon) = \epsilon > 0$
$\mu(C.c.r) = 0$

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\[ \mu(C_3) = 0 \]
Continued Radicals and Cantor Sets

- \( \mu(C_3) = 0 \)
- \( \mu(C_\varepsilon) = \varepsilon > 0 \)
$\mu(C_3) = 0$

$\mu(C_\varepsilon) = \varepsilon > 0$

$\mu(C_{c.r.}) = 0$
Why study these?
Why study these?

**Theorem**

*Any real number can be represented as a continued radical*

\[ \sqrt{a_0} + \sqrt{a_1} + \ldots \text{ where } a_i \in \mathbb{Z} \text{ and for } i \geq 1, a_i \text{ is } 0, 1, \text{ or } 2. \text{ (Sizer 1986)} \]
Can information and results obtained from continued fractions be applied to continued radicals?
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Thickness

Questions